Spherically symmetric solutions in $f(R)$ gravity *via* Noether Symmetry Approach

S. Capozziello[1], **A. Stabile**[2], **A. Troisi**[1]

[1]Dipartimento di Scienze Fisiche, Università di Napoli "Federico II", and INFN, Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126 - Napoli, Italy

[2]Dipartimento di Fisica "E.R. Caianiello" and INFN, Sez. di Napoli, Università di Salerno, via S. Allende, I-84081 - Baronissi (Sa), Italy

Abstract. We search for spherically symmetric solutions of $f(R)$ theories of gravity via the Noether Symmetry Approach. A general formalism in the metric framework is developed considering a point-like $f(R)$-Lagrangian where spherical symmetry is required. Examples of exact solutions are given.

PACS numbers: 98.80.-k, 95.35.+x, 95.35.+d, 04.50.+h

1. Introduction

Extended Theories of Gravity have become a sort of paradigm in modern physics since they seem to solve several shortcomings of standard General Relativity (GR) related to cosmology, astrophysics and quantum field theory. The idea to extend Einstein's theory of gravitation is fruitful and economic with respect to several attempts which try to solve problems by adding new and, most of times, unjustified new ingredients in order to give a self-consistent picture of dynamics. The today observed accelerated expansion of Hubble flow and the missing matter of astrophysical large scale structures, are primarily enclosed in these considerations. Both the issues could be solved changing the gravitational sector, i.e. the *l.h.s.* of field equations. The philosophy is alternative to add new cosmic fluids (new components in the *r.h.s.* of field equations) which should give rise to clustered structures (dark matter) or to accelerated dynamics (dark energy) thanks to exotic equations of state. In particular, relaxing the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar R, like in the Hilbert-Einstein formulation, one can take into account, as a minimal extension, an effective action where the gravitational Lagrangian is a generic $f(R)$ function. As further request, one can ask for $f(R)$ being analytical in order to recover, at least locally, the positive results of GR. Several studies in this sense show that cosmic dynamics at early [1, 2] and late [3] epochs can be successfully reproduced. On the other hand, flat rotation curves of spiral galaxies can be fitted adopting the low energy limit of power law $f(R)$ and without considering huge amounts of dark matter in galactic haloes [4, 5].

Interesting indications have been achieved also for other $f(R)$ functions [6] and for large scale structure (this issue is still matter of debate [7], the CMBR - spectrum turns out to be very slightly affected if the Lagrangian shows a small deviation from the standard Hilbert-Einstein form [8]). Despite of these positive results, the problem is still open since the degeneration of viable Lagrangians has not been removed yet [9] and a final theory embracing the phenomenology at local and large scales or at early and late epochs, considering only the $f(R)$ approach, is not available up to now (see [10, 11] for a recent discussion). On the other hand, Solar System experiments are not giving univocal constraints on the Parametrized Post-Newtonian limit of such theories: some authors claim for the fact that there is room for Extended Theories of Gravity considering experimental data [12, 13, 14], others claim for ruling out such theories with respect to GR [15, 16]. The debate essentially lies on the physical meaning of the conformal transformations. For a recent and illuminating discussion on the argument see [17] and references therein. A part the controversies, and the open discussions, several efforts have been done to give self-consistent formulations of $f(R)$ gravity [18] and several approaches have been pursued to find out solutions of the field equations coming out from such theories, both in metric and in Palatini formalism.

In a recent paper [19], spherically symmetric solutions for $f(R)$ gravity in vacuum have been found considering relations among functions defining the spherical metric or imposing a constant Ricci curvature scalar. The authors have been able to reconstruct, at the end, the form of some $f(R)$ theories, discussing their physical relevance. In [20], the same authors have discussed static spherically symmetric perfect fluid solutions for $f(R)$ gravity in metric formalism. They showed that a given matter distribution is not capable of determining the functional form of $f(R)$.

In this paper, we want to seek for a general method to find out spherically symmetric solutions in $f(R)$ gravity and, eventually, in generic extended theories of gravity. Asking for a certain symmetry of the metric, we would like to investigate if such a symmetry holds for a generic theory of gravity. In particular for the $f(R)$ theories. Specifically, we want to apply the Noether Symmetry Approach [22] in order to search for spherically symmetric solutions in generic $f(R)$ theories of gravity. This means that we consider the spherical symmetry for the metric as a Noether symmetry and search for $f(R)$ Lagrangians compatible with it. The method can give several hints toward the formulation of Birkhoff's theorem (see [23] for a general formulation) for these theories since, up to now, there are controversial results in this direction‡.

The layout of the paper is the following. In Sec.2, we derive and discuss the field equations for $f(R)$ gravity. Sec.3 is devoted to the construction of the point-like Lagrangian for a generic $f(R)$ theory. We point out that imposing the spherical symmetry in the action, and then deriving the Euler-Lagrange equations, is equivalent to derive first the field equations and then to impose the spherical symmetry. This

‡ Some authors state that the theorem is not valid in general [24] while others claims for its validity for specific classes of $f(R)$ [25, 26, 27].

procedure§ allows to construct a Noether vector. The Noether Symmetry Approach to reduce dynamics is described in Sec.4. The goal of the method is to construct a vector field which, contracted with the point-like Lagrangian, allows to find out, if existing, the conserved quantities of dynamics. Then it is possible to recast the original Lagrangian in a new set of variables where cyclic ones explicitly appear. The number of cyclic variables is equal to the number of conserved quantities. This technique reduces the order of derivation of the equations and simplifies the process to achieve exact solutions. We carry out the method for the point-like $f(R)$ Lagrangian in spherical symmetry and find out some exact solutions in Sec.5. Discussion and conclusions are drawn in Sec.6. Appendix A is devoted to the field equations in spherical symmetry for a generic $f(R)$. In Appendix B, we write explicitly the PDE system, derived from the contraction $L_X\mathcal{L} = 0$, which is the existence condition for the Noether Symmetry.

2. The $f(R)$ gravity action and the field equations

The action

$$\mathcal{A} = \int d^4x\sqrt{-g}\,[f(R) + \mathcal{L}_m]\,, \tag{1}$$

describes a theory of gravity where $f(R)$ is a generic function of scalar curvature, g is the determinant of the metric tensor and $\mathcal{L}_m$ is the standard fluid matter minimally coupled with gravity. We are assuming physical units $8\pi G = 1$. The field equations in the metric approach (i.e. obtained by a variation with respect to the metric $g_{\mu\nu}$) are

$$f_R(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f_R(R)_{;\mu\nu} + g_{\mu\nu}\Box f_R(R) \;=\; T^m_{\mu\nu}\,, \tag{2}$$

where $f_R(R) = \dfrac{df(R)}{dR}$. Eqs.(2) are of fourth order due to the covariant derivatives of $f_R(R)$ and reduce to the standard Einstein ones if $f(R) = R$. $T^m_{\mu\nu}$ is the matter fluid stress-energy tensor. Defining a *curvature stress - energy tensor*

$$T^{curv}_{\mu\nu} \;=\; \frac{1}{f_R(R)}\left\{\frac{1}{2}g_{\mu\nu}\,[f(R) - Rf_R(R)] + f_R(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\mu\nu}g_{\alpha\beta})\right\}, \tag{3}$$

Eqs.(2) can be recast in the Einstein - like form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{curv}_{\mu\nu} + T^m_{\mu\nu}/f_R(R) \tag{4}$$

where matter non - minimally couples to geometry through the term $1/f_R(R)$. In this paper, we shall seek for exact solutions in vacuum so that we can assume $T^m_{\mu\nu} = 0$.

By contracting with respect to the metric tensor, one can reveal the analogy of $f_R(R)$ with a scalar field being the trace equation

$$3\Box f_R(R) + Rf_R(R) - 2f(R) \;=\; 0\,. \tag{5}$$

§ It is straightforward to show that this method works also for Friedmann-Robertson-Walker (FRW) metric or generic Bianchi's metrics and it is extremely useful to find out cosmological solutions [21, 22, 29, 30].

It can be seen as a Klein-Gordon equation for an effective scalar field if the identifications

$$\varphi \longrightarrow f_R(R)\,, \qquad \frac{dV(\varphi)}{d\varphi} \longrightarrow \frac{Rf_R(R)-2f(R)}{3}\,,$$

are considered [1].

3. The point-like $f(R)$ Lagrangian in spherical symmetry

As hinted in the introduction, the aim of this paper is to work out an approach to obtain spherically symmetric solutions in fourth order gravity by means of Noether Symmetries. In order to develop this approach (the method will be outlined in Sec.4), we need to deduce a point-like Lagrangian from the action (1). Such a Lagrangian can be obtained by imposing the spherical symmetry in the field action (1). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier defined by assuming the Ricci scalar, argument of the function $f(R)$ in spherical symmetry. Elsewhere, this approach has been successfully used for the FRW metric with the purpose to find out cosmological solutions [22, 29, 31].
In general, a spherically symmetric spacetime can be described assuming the metric:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - M(r)d\Omega\,, \tag{6}$$

where $d\Omega = d\theta^2 + \sin\theta^2 d\varphi^2$ is the angular element. Obviously the conditions $M(r) = r^2$ and $B(r) = A^{-1}(r)$ are requested to obtain the standard Schwarzschild case of GR. This condition is necessary if one wants to recover the standard measure of a circumference when r is the radius of a circle. Our goal is to reduce the field action (1) to a form with a finite degrees of freedom, that is the canonical action

$$\mathcal{A} = \int dr \mathcal{L}(A, A', B, B', M, M', R, R') \tag{7}$$

where the Ricci scalar R and the potentials A, B, M are the set of independent variables defining the configuration space. Prime indicates the derivative with respect to the radial coordinate r. In order to achieve the point-like Lagrangian in this set of coordinates, we write

$$\mathcal{A} = \int d^4x\sqrt{-g}\Big[f(R) - \lambda(R - \bar{R})\Big]\,, \tag{8}$$

where λ is the Lagrangian multiplier and $\bar{R}$ is the Ricci scalar expressed in terms of the metric (6)

$$\bar{R} = \frac{A''}{AB} + 2\frac{M''}{BM} + \frac{A'M'}{ABM} - \frac{A'^2}{2A^2B} - \frac{M'^2}{2BM^2} - \frac{A'B'}{2AB^2} - \frac{B'M'}{B^2M} - \frac{2}{M}\,, \tag{9}$$

which can be recast in the more compact form

$$\bar{R} = R^* + \frac{A''}{AB} + 2\frac{M''}{BM}\,, \tag{10}$$

where R^* collects first order derivative terms. The Lagrange multiplier λ is obtained by varying the action (8) with respect to R. One gets $\lambda = f_R(R)$. By expressing the determinant g and $\bar{R}$ in terms of A, B and M, we have, from Eq.(8),

$$\mathcal{A} = \int dr A^{1/2}B^{1/2}M\Big[f - f_R\Big(R - R^* - \frac{A''}{AB} - 2\frac{M''}{BM}\Big)\Big] = \tag{11}$$
$$= \int dr\Big\{A^{1/2}B^{1/2}M\Big[f - f_R(R - R^*)\Big] - \Big(\frac{f_R M}{A^{1/2}B^{1/2}}\Big)'A' - 2\Big(\frac{A^{1/2}}{B^{1/2}}f_R\Big)'M'\Big\}.$$

The two lines differs for a divergence term which we discard integrating by parts. Therefore, the point-like Lagrangian becomes:

$$\mathcal{L} = -\frac{A^{1/2}f_R}{2MB^{1/2}}M'^2 - \frac{f_R}{A^{1/2}B^{1/2}}A'M' - \frac{Mf_{RR}}{A^{1/2}B^{1/2}}A'R' +$$
$$-\frac{2A^{1/2}f_{RR}}{B^{1/2}}R'M' - A^{1/2}B^{1/2}[(2+MR)f_R - Mf]\,, \tag{12}$$

which is canonical since only the configuration variables and their first order derivatives with respect to r are present. Eq. (12) can be recast in more compact form introducing the matrix formalism:

$$\mathcal{L} = \underline{q}'^t\hat{T}\underline{q}' + V \tag{13}$$

where $\underline{q} = (A, B, M, R)$ and $\underline{q}' = (A', B', M', R')$ are the generalized positions and velocities associated to $\mathcal{L}$. The index "t" indicates the transposed column vector. The kinetic tensor is given by $\hat{T}_{ij} = \frac{\partial^2\mathcal{L}}{\partial q'_i\partial q'_j}$. $V = V(q)$ is the potential depending only on the configuration variables. The Euler - Lagrange equations read

$$\frac{d}{dr}\nabla_{q'}\mathcal{L} - \nabla_q\mathcal{L} = 2\frac{d}{dr}\Big(\hat{T}\underline{q}'\Big) - \nabla_q V - \underline{q}'^t\Big(\nabla_q\hat{T}\Big)\underline{q}' =$$
$$= 2\hat{T}\underline{q}'' + 2\Big(\underline{q}'\cdot\nabla_q\hat{T}\Big)\underline{q}' - \nabla_q V - \underline{q}'^t\Big(\nabla_q\hat{T}\Big)\underline{q}' = 0 \tag{14}$$

which furnish the equations of motion in term of A, B, M and R, respectively. The field equation for R corresponds to the constraint among the configuration coordinates. It is worth noting that the Hessian determinant of (12), $\Big|\Big|\frac{\partial^2\mathcal{L}}{\partial q'_i\partial q'_j}\Big|\Big|$, is zero. This result clearly depends on the absence of the generalized velocity B' into the point - like Lagrangian. As matter of fact, using a point-like Lagrangian approach implies that the metric variable B does not contributes to dynamics, but the equation of motion for B has to be considered as a further constraint equation.

Beside the Euler - Lagrange equations (14), one has to take into account the energy $E_{\mathcal{L}}$:

$$E_{\mathcal{L}} = \underline{q}'\cdot\nabla_{q'}\mathcal{L} - \mathcal{L} \tag{15}$$

which can be easily recognized to be coincident with the Euler-Lagrangian equation for the component B of the generalized position $\underline{q}$. Then the Lagrangian (12) has three degrees of freedom and not four, as we would expected "a priori".

Now, since the motion equation describing the evolution of the metric potential B does not depends on its derivative, it can be explicitly solved in term of B as a function of other coordinates:

$$B=\frac{2M^2f_{RR}A'R'+2Mf_RA'M'+4AMf_{RR}M'R'+Af_RM'^2}{2AM[(2+MR)f_R-Mf]}\,. \tag{16}$$

By inserting Eq.(16) into the Lagrangian (12), we obtain a non-vanishing Hessian matrix removing the singular dynamics. The new Lagrangian reads‖

$$\mathcal{L}^*=\mathbf{L}^{1/2} \tag{17}$$

with

$$\mathbf{L}=\underline{q'}^t\hat{\mathbf{L}}\underline{q'}=\frac{[(2+MR)f_R-fM]}{M}\times$$

$$\times[2M^2f_{RR}A'R'+2MM'(f_RA'+2Af_{RR}R')+Af_RM'^2]\,. \tag{18}$$

Since $\dfrac{\partial\mathbf{L}}{\partial r}=0$, $\mathbf{L}$ is canonical ($\mathbf{L}$ is the quadratic form of generalized velocities, A', M' and R' and then coincides with the Hamiltonian), so that we can consider $\mathbf{L}$ as the new Lagrangian with three degrees of freedom. The crucial point of such a replacement is that the Hessian determinant is now non - vanishing, being:

$$\left|\left|\frac{\partial^2\mathbf{L}}{\partial q'_i\partial q'_j}\right|\right|=3AM[(2+MR)f_R-Mf]^3f_Rf_{RR}{}^2\,. \tag{19}$$

Obviously, we are supposing that $(2+MR)f_R-Mf\neq 0$, otherwise the above definitions of B, [Eq.(16)], and $\mathbf{L}$, [Eq.(18)], lose of significance, besides we assume $f_{RR}\neq 0$ to admit a wide class of fourth order gravity models. The case $f(R)=R$ requires a different investigation. In fact, considering the GR point - like Lagrangian needs a further lowering of degrees of freedom of the system and the previous results cannot be straightforwardly considered. From (12), we get:

$$\mathcal{L}_{GR}=-\frac{A^{1/2}}{2MB^{1/2}}M'^2-\frac{1}{A^{1/2}B^{1/2}}A'M'-2A^{1/2}B^{1/2}\,, \tag{20}$$

whose Euler-Lagrange equations provide the standard equations of GR for Schwarzschild metric. It is easy to see the absence of the generalized velocity B' in Eq.(20). Again, the Hessian determinant is zero. Nevertheless, considering, as above, the constraint (16) for B, it is possible to obtain a Lagrangian with a non-vanishing Hessian. In particular one has:

$$B_{GR}=\frac{M'^2}{4M}+\frac{A'M'}{2A}\,, \tag{21}$$

$$\mathcal{L}^*_{GR}=\mathbf{L}^{1/2}_{GR}=\sqrt{\frac{M'(2MA'+AM')}{M}}\,, \tag{22}$$

‖ Lowering the dimension of configuration space through the substitution (16) does not affect the dynamics, since B is a non-evolving quantity. In fact, introducing Eq. (16) directly into the dynamical equations given by (12), they coincide with those derived by (18).

and then the Hessian determinant is

$$\left|\left|\frac{\partial^2 \mathbf{L}_{GR}}{\partial q'_i \partial q'_j}\right|\right| = -1\,, \tag{23}$$

which is nothing else but a non-vanishing sub-matrix of the $f(R)$ Hessian matrix.
Considering the Euler - Lagrange equations coming from (21) and (22), one obtains the vacuum solutions of GR, that is:

$$A = k_4 - \frac{k_3}{r + k_1}\,, \qquad B = \frac{k_2 k_4}{A}\,, \qquad M = k_2(r + k_1)^2\,. \tag{24}$$

In particular, the standard form of Schwarzschild solution is obtained for $k_1 = 0$, $k_2 = 1$, $k_3 = \dfrac{2GM}{c^2}$ and $k_4 = 1$.
A formal summary of the field equations descending from the point - like Lagrangian and their relation with respect to the ones of the standard approach is given in Tab.1.

Field equations approach		Point-like Lagrangian approach
$\downarrow$		$\downarrow$
$\delta\int d^4x\sqrt{-g}f = 0$	$\leftrightarrows$	$\delta\int dr\mathcal{L} = 0$
$\downarrow$		$\downarrow$
$H_{\mu\nu} = \partial_\rho\frac{\partial(\sqrt{-g}f)}{\partial_\rho g^{\mu\nu}} - \frac{\partial(\sqrt{-g}f)}{\partial g^{\mu\nu}} = 0$		$\frac{d}{dr}\nabla_{q'}\mathcal{L} - \nabla_q\mathcal{L} = 0$
	$\leftrightarrows$	
$H = g^{\mu\nu}H_{\mu\nu} = 0$		$E_{\mathcal{L}} = \underline{q}' \cdot \nabla_{q'}\mathcal{L} - \mathcal{L}$
$\downarrow$		$\downarrow$
$H_{00} = 0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial A'} - \frac{\partial\mathcal{L}}{\partial A} = 0$
$H_{rr} = 0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial B'} - \frac{\partial\mathcal{L}}{\partial B} \propto E_{\mathcal{L}} = 0$
$H_{\theta\theta} = \csc^2\theta H_{\varphi\varphi} = 0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial M'} - \frac{\partial\mathcal{L}}{\partial M} = 0$
$H = A^{-1}H_{00} - B^{-1}H_{rr} - 2M^{-1}\csc^2\theta H_{\varphi\varphi} = 0$	$\leftrightarrows$	A combination of the above equations

Table 1. The field-equations approach and the point-like Lagrangian approach differ since the symmetry, in our case the spherical one, can be imposed whether in the field equations, after standard variation with respect to the metric, or directly into the Lagrangian, which becomes point-like. The energy $E_{\mathcal{L}}$ corresponds to the 00-component of $H_{\mu\nu}$. The absence of B' in the Lagrangian implies the proportionality between the constraint equation for B and the energy function $E_{\mathcal{L}}$. As a consequence, the number of independent equations is three (as the number of unknown functions). Finally it is obvious the correspondence between $\theta\theta$ component and field equation for M. The explicit form of field equations $H_{\mu\nu}$ is given in App.B.

4. The Noether Symmetry Approach

In order to find out solutions for the Lagrangian (18), we can search for symmetries related to cyclic variables and then reduce dynamics. This approach allows, in principle, to select $f(R)$ gravity models compatible with spherical symmetry. As a general remark, the Noether Theorem states that conserved quantities are related to the existence of cyclic variables into dynamics [32, 33, 34]. Let us give a summary of the approach for finite dimensional dynamical systems.

Let $l(q^i, \dot{q}^i)$ be a canonical, non-degenerate point-like Lagrangian where

$$\frac{\partial l}{\partial \lambda} = 0\,; \qquad \det H_{ij} \stackrel{\text{def}}{=} \det \left|\left| \frac{\partial^2 l}{\partial \dot{q}^i \partial \dot{q}^j} \right|\right| \neq 0\,, \tag{25}$$

with H_{ij} as above, the Hessian matrix related to l. The dot indicates derivatives with respect to the affine parameter λ which, ordinarily, corresponds to time t. In our case, it is the radial coordinate r. In standard problems of analytical mechanics, l is in the form

$$l = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})\;, \tag{26}$$

where T and V are the "kinetic" and "potential energy" respectively. T is a positive definite quadratic form in $\dot{\mathbf{q}}$. The energy function associated with l is

$$E_l \equiv \frac{\partial l}{\partial \dot{q}^i}\dot{q}^i - l\,, \tag{27}$$

which is the total energy $T + V$. It has to be noted that E_l is, in any case, a constant of motion. In this formalism, we are going to consider only transformations which are point-transformations. Any invertible and smooth transformation of the "positions" $Q^i = Q^i(\mathbf{q})$ induces a transformation of the "velocities" such that

$$\dot{Q}^i(\mathbf{q}) = \frac{\partial Q^i}{\partial q^j}\dot{q}^j\;; \tag{28}$$

the matrix $\mathcal{J} = ||\partial Q^i/\partial q^j||$ is the Jacobian of the transformation on the positions, and it is assumed to be nonzero. The Jacobian $\widetilde{\mathcal{J}}$ of the "induced" transformation is easily derived and $\mathcal{J} \neq 0 \rightarrow \widetilde{\mathcal{J}} \neq 0$. Usually, this condition is not satisfied in the whole space but only in the neighbor of a point. It is *local transformation*. If one extends the transformation to the maximal submanifold such that $\mathcal{J} \neq 0$, it is possible to get troubles for the whole manifold due to possible different topologies [34].

A point transformation $Q^i = Q^i(\mathbf{q})$ can depend on one (or more than one) parameter. Let us assume that a point transformation depends on a parameter ε, i.e. $Q^i = Q^i(\mathbf{q}, \varepsilon)$, and that it gives rise to a one–parameter Lie group. For infinitesimal values of ε, the transformation is then generated by a vector field: for instance, as well known, $\partial/\partial x$ represents a translation along the x axis, $x(\partial/\partial y) - y(\partial/\partial x)$ is a rotation around the z axis and so on. In general, an infinitesimal point transformation is represented by a generic vector field on Q

$$\mathbf{X} = \alpha^i(\mathbf{q})\frac{\partial}{\partial q^i}\;. \tag{29}$$

The induced transformation (28) is then represented by

$$\mathbf{X}^c = \alpha^i(\mathbf{q})\frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda}\alpha^i(\mathbf{q})\right)\frac{\partial}{\partial \dot{q}^j}\;. \tag{30}$$

$\mathbf{X}^c$ is called the "complete lift" of $\mathbf{X}$ [34]. A function $f(\mathbf{q},\dot{\mathbf{q}})$ is invariant under the transformation $\mathbf{X}^c$ if

$$L_{\mathbf{X}^c} f \stackrel{\text{def}}{=} \alpha^i(\mathbf{q})\frac{\partial f}{\partial q^i} + \left(\frac{d}{d\lambda}\alpha^i(\mathbf{q})\right)\frac{\partial f}{\partial \dot{q}^j} \;=\; 0\,, \tag{31}$$

where $L_{\mathbf{X}^c} f$ is the Lie derivative of f. In particular, if $L_{\mathbf{X}^c} l = 0$, $\mathbf{X}^c$ is said to be a *symmetry* for the dynamics derived by l.

In order to see how Noether's theorem and cyclic variables are related, let us consider a Lagrangian l and its Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial l}{\partial \dot{q}^j} - \frac{\partial l}{\partial q^j} = 0\,. \tag{32}$$

Let us consider also the vector field (30). Contracting (32) with the α^i's gives

$$\alpha^j\left(\frac{d}{d\lambda}\frac{\partial l}{\partial \dot{q}^j} - \frac{\partial l}{\partial q^j}\right) = 0\,. \tag{33}$$

Being

$$\alpha^j\frac{d}{d\lambda}\frac{\partial l}{\partial \dot{q}^j} = \frac{d}{d\lambda}\left(\alpha^j\frac{\partial l}{\partial \dot{q}^j}\right) - \left(\frac{d\alpha^j}{d\lambda}\right)\frac{\partial l}{\partial \dot{q}^j}\,, \tag{34}$$

from (33), we obtain

$$\frac{d}{d\lambda}\left(\alpha^i\frac{\partial l}{\partial \dot{q}^i}\right) = L_{\mathbf{X}} l\,. \tag{35}$$

The immediate consequence is the *Noether Theorem*¶:

If $L_{\mathbf{X}} l = 0$, then the function

$$\Sigma_0 = \alpha^i\frac{\partial l}{\partial \dot{q}^i}\,, \tag{36}$$

is a constant of motion.

Remark. Eq.(36) can be expressed independently of coordinates as a contraction of $\mathbf{X}$ by a Cartan one–form

$$\theta_l \stackrel{\text{def}}{=} \frac{\partial l}{\partial \dot{q}^i}dq^i\,. \tag{37}$$

For a generic vector field $\mathbf{Y} = y^i\partial/\partial x^i$, and one–form $\beta = \beta_i dx^i$, we have, by definition, $i_{\mathbf{Y}}\beta = y^i\beta_i$. Thus Eq.(36) can be written as

$$i_{\mathbf{X}}\theta_l = \Sigma_0\,. \tag{38}$$

By a point–transformation, the vector field $\mathbf{X}$ becomes

$$\widetilde{\mathbf{X}} = (i_{\mathbf{X}}dQ^k)\frac{\partial}{\partial Q^k} + \left(\frac{d}{d\lambda}(i_x dQ^k)\right)\frac{\partial}{\partial \dot{Q}^k}\,. \tag{39}$$

¶ In the following, with abuse of notation, we shall write $\mathbf{X}$ instead of $\mathbf{X}^c$, whenever no confusion is possible.

We see that $\widetilde{\mathbf{X}}'$ is still the lift of a vector field defined on the "space of positions". If $\mathbf{X}$ is a symmetry and we choose a point transformation such that

$$i_{\mathbf{X}}dQ^1 = 1\,;\quad i_{\mathbf{X}}dQ^i = 0 \quad i \neq 1\,, \tag{40}$$

we get

$$\widetilde{\mathbf{X}} = \frac{\partial}{\partial Q^1}\,;\qquad \frac{\partial l}{\partial Q^1} = 0\,. \tag{41}$$

Thus Q^1 is a cyclic coordinate and the dynamics can be reduced [32, 33].
Remarks:

(i) The change of coordinates defined by (40) is not unique. Usually a clever choice is very important.

(ii) In general, the solution of Eq.(40) is not well defined on the whole space. It is *local* in the sense explained above.

(iii) It is possible that more than one $\mathbf{X}$ is found, say for instance $\mathbf{X}_1$, $\mathbf{X}_2$. If they commute, i.e. $[\mathbf{X}_1, \mathbf{X}_2] = 0$, then it is possible to obtain two cyclic coordinates by solving the system

$$i_{\mathbf{X_1}}dQ^1 = 1;\, i_{\mathbf{X_2}}dQ^2 = 1;\, i_{\mathbf{X_1}}dQ^i = 0;\, i \neq 1;\, i_{\mathbf{X_2}}dQ^i = 0;\, i \neq 2\,. \tag{42}$$

The transformed fields will be $\partial/\partial Q^1$, $\partial/\partial Q^2$. If they do not commute, this procedure is clearly not applicable, since commutation relations are preserved by diffeomorphisms. Let us note that $\mathbf{X}_3 = [\mathbf{X}_1, \mathbf{X}_2]$ is also a symmetry, indeed, being $L_{\mathbf{X_3}}l = L_{\mathbf{X_1}}L_{\mathbf{X_2}}l - L_{\mathbf{X_2}}L_{\mathbf{X_1}}l = 0$. If $\mathbf{X_3}$ is independent of $\mathbf{X_1}$, $\mathbf{X_2}$, we can go on until the vector fields close the Lie algebra. The usual way to treat this situation is to make a Legendre transformation, going to the Hamiltonian formalism and to a Lie algebra of Poisson brackets. If we look for a reduction with cyclic coordinates, this procedure is possible in the following way:

- we arbitrarily choose one of the symmetries, or a linear combination of them, and get new coordinates as above. After the reduction, we get a new Lagrangian $\widetilde{l}(\mathbf{Q})$;
- we search again for symmetries in this new space, make a new reduction and so on until possible;
- if the search fails, we try again with another of the existing symmetries.

Let us now assume that l is of the form (26). As $\mathbf{X}$ is of the form (30), $L_{\mathbf{X}}l$ will be a homogeneous polynomial of second degree in the velocities plus a inhomogeneous term in the q^i. Since such a polynomial has to be identically zero, each coefficient must be independently zero. If n is the dimension of the configuration space, we get $\{1 + n(n+1)/2\}$ partial differential equations (PDE). The system is overdetermined, therefore, if any solution exists, it will be expressed in terms of integration constants instead of boundary conditions. It is also obvious that an overall constant factor in the Lie vector $\mathbf{X}$ is irrelevant. In other words, the Noether Symmetry Approach can be used to select functions which assign the models and, as we shall see below, such functions (and then the models) can be physically relevant. This fact justifies the method at least *a posteriori*.

5. The Noether Approach for $f(R)$ gravity in spherical symmetry

Since the above considerations, if one assumes the spherical symmetry, the role of the *affine parameter* is played by the coordinate radius r. In this case, the configuration space is given by $\mathcal{Q} = \{A, M, R\}$ and the tangent space by $\mathcal{TQ} = \{A, A', M, M', R, R'\}$. On the other hand, according to the Noether theorem, the existence of a symmetry for dynamics described by the Lagrangian (18) implies a constant of motion. Let us apply the Lie derivative to the (18), we have[+] :

$$L_{\mathbf{X}}\mathbf{L} \;=\; \underline{\alpha}\cdot\nabla_q\mathbf{L} + \underline{\alpha}'\cdot\nabla_{q'}\mathbf{L} = \underline{q}'^t\Big[\underline{\alpha}\cdot\nabla_q\hat{\mathbf{L}} + 2\Big(\nabla_q\alpha\Big)^t\hat{\mathbf{L}}\Big]\underline{q}'\,, \tag{43}$$

which vanish if the functions $\underline{\alpha}$ satisfy the following system (see App.B for details)

$$\underline{\alpha}\cdot\nabla_q\hat{\mathbf{L}} + 2(\nabla_q\underline{\alpha})^t\hat{\mathbf{L}} \;=\; 0 \;\longrightarrow\; \qquad \alpha_i\frac{\partial\hat{\mathbf{L}}_{km}}{\partial q_i} + 2\frac{\partial\alpha_i}{\partial q_k}\hat{\mathbf{L}}_{im} = 0\,. \tag{44}$$

Solving the system (44) means to find out the functions α_i which assign the Noether vector. However the system (44) implicitly depends on the form of $f(R)$ and then, by solving it, we get also $f(R)$ theories compatible with spherical symmetry. On the other hand, by choosing the $f(R)$ form, we can explicitly solve (44). As an example, one finds that the system (44) is satisfied if we chose

$$f(R) \;=\; f_0R^s \qquad \underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = \Big((3-2s)kA,\; -kM,\; kR\Big) \tag{45}$$

with s a real number, k an integration constant and f_0 a dimensional coupling constant[*]. This means that for any $f(R) = R^s$ exists, at least, a Noether symmetry and a related constant of motion Σ_0 :

$$\Sigma_0 \;=\; \underline{\alpha}\cdot\nabla_{q'}\mathbf{L} =$$

$$= 2skMR^{2s-3}[2s + (s-1)MR][(s-2)RA' - (2s^2 - 3s + 1)AR']\,. \tag{46}$$

A physical interpretation of Σ_0 is possible if one gives an interpretation of this quantity in GR. In such a case, with $s = 1$, the above procedure has to be applied to the Lagrangian (22). We obtain the solution

$$\underline{\alpha}_{GR} = (-kA,\; kM)\,. \tag{47}$$

The functions A and M give the Schwarzschild solution (24), and then the constant of motion acquires the form

$$\Sigma_0 = \frac{2GM}{c^2}\,. \tag{48}$$

[+] From now on, $\underline{q}$ indicates the vector (A, M, R).

[*] The dimensions are given by R^{1-s} in term of the Ricci scalar. For the sake of simplicity we will put $f_0 = 1$ in the forthcoming discussion.

In other words, in the case of Einstein gravity, the Noether symmetry gives as a conserved quantity the Schwarzschild radius or the mass of the gravitating system.

Another solution can be find out for $R = R_0$ where R_0 is a constant (see also [19]). In this case, the field equations (2) reduce to

$$R_{\mu\nu} + k_0 g_{\mu\nu} = 0\,, \tag{49}$$

where $k_0 = -\frac{1}{2} f(R_0)/f_R(R_0)$. The general solution is

$$A(r) = \frac{1}{B(r)} = 1 + \frac{k_0}{r} + \frac{R_0}{12} r^2\,, \qquad M = r^2 \tag{50}$$

with the special case

$$A(r) = \frac{1}{B(r)} = 1 + \frac{k_0}{r}\,, \qquad M = r^2\,, \qquad R = 0\,. \tag{51}$$

The solution (50) is the well known Schwarzschild-de Sitter one which is a solution in most of modified gravity theories. It evades the Solar System constraints due to the smallness of the effective cosmological constant. However, other spherically symmetric solutions, different from this, are more significant for Solar System tests.

In the general case $f(R) = R^s$, the Lagrangian (18) becomes

$$\mathbf{L} = \frac{sR^{2s-3}[2s + (s-1)MR]}{M} \times$$

$$\times [2(s-1)M^2 A' R' + 2MRM'A' + 4(s-1)AMM'R' + ARM'^2]\,, \tag{52}$$

and the expression (16) for B is

$$B = \frac{s[2(s-1)M^2 A' R' + 2MRM'A' + 4(s-1)AMM'R' + ARM'^2]}{2AMR[2s + (s-1)MR]} \tag{53}$$

As it can be easily checked, GR is recovered when $s = 1$.

Using the constant of motion (46), we solve in term of A and obtain

$$A = R^{\frac{2s^2-3s+1}{s-2}} \left\{ k_1 + \Sigma_0 \int \frac{R^{\frac{4s^2-9s+5}{2-s}} dr}{2ks(s-2)M[2s + (s-1)MR]} \right\} \tag{54}$$

for $s \neq 2$, with k_1 an integration constant. For $s = 2$, one finds

$$A = -\frac{\Sigma_0}{12kr^2(4 + r^2 R)RR'}\,. \tag{55}$$

These relations allow to find out general solutions for the field equations giving the dependence of the Ricci scalar on the radial coordinate r. For example, a solution is found for

$$s = 5/4\,, \quad M = r^2\,, \quad R = 5r^{-2}\,, \tag{56}$$

obtaining the spherically symmetric metric

$$ds^2 = \frac{1}{\sqrt{5}}(k_2 + k_1 r)dt^2 - \frac{1}{2}\left(\frac{1}{1 + \frac{k_2}{k_1 r}}\right) dr^2 - r^2 d\Omega\,, \tag{57}$$

with $k_2 = \frac{32\Sigma_0}{225k}$. It is worth noting that such exact solution is in the range of s values ruled out by Solar System observations, as pointed out in [28].

6. Discussion and Conclusions

In this paper, we have discussed a general method to find out exact solutions in Extended Theories of Gravity when a spherically symmetric background is taken into account. In particular, we have searched for exact spherically symmetric solutions in $f(R)$ gravity by asking for the existence of Noether symmetries. We have developed a general formalism and given some examples of exact solutions. The procedure consists in: *i*) considering the point-like $f(R)$ Lagrangian where spherical symmetry has been imposed; *ii*) deriving the Euler-Lagrange equations; *iii*) searching for a Noether vector field; *iv*) reducing dynamics and then integrating the equations of motion using conserved quantities. Viceversa, the approach allows also to select families of $f(R)$ models where a particular symmetry (in this case the spherical one) is present. As examples, we discussed power law models and models with constant Ricci curvature scalar. However, the above method can be further generalized. If a symmetry exists, the Noether Approach allows, as discussed in Sec.4, transformations of variables where the cyclic ones are evident. This fact allows to reduce dynamics and then to get more easily exact solutions. For example, since we know that $f(R) = R^s$ - gravity admit a conserved quantity, a coordinate transformation can be induced by the Noether symmetry. We ask for the coordinate transformation :

$$\mathbf{L} = \mathbf{L}(\underline{q}, \underline{q}') = \mathbf{L}(A, M, R, A', M', R') \rightarrow \widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}(\widetilde{M}, \widetilde{R}, \widetilde{A}', \widetilde{M}', \bar{R}') \,, \tag{58}$$

for the Lagrangian (18), where the Noether symmetry, and then the conserved quantity, corresponds to the cyclic variable $\widetilde{A}$. If more than one symmetry exists, one can find more than one cyclic variables. In our case, if three Noether symmetries exist, we can transform the Lagrangian $\mathbf{L}$ in a Lagrangian with three cyclic coordinates, that is $\widetilde{A} = \widetilde{A}(\underline{q})$, $\widetilde{M} = \widetilde{M}(\underline{q})$ and $\widetilde{R} = \widetilde{R}(\underline{q})$ which are function of the old ones. These new functions have to satisfy the following system

$$\begin{cases} (3-2s)A\frac{\partial \widetilde{A}}{\partial A} - M\frac{\partial \widetilde{A}}{\partial M} + R\frac{\partial \widetilde{A}}{\partial R} = 1 \,, \\ \\ (3-2s)A\frac{\partial \widetilde{q}_i}{\partial A} - M\frac{\partial \widetilde{q}_i}{\partial M} + R\frac{\partial \widetilde{q}_i}{\partial R} \; = \; 0 \,, \end{cases} \tag{59}$$

with $i = 2, 3$ (we have put $k = 1$). A solution of (59) is given by the set (for $s \neq 3/2$)

$$\begin{cases} \widetilde{A} = \frac{\ln A}{(3-2s)} + F_A(A^{\frac{\eta_A}{3-2s}} M^{\eta_A}, A^{\frac{\xi_A}{2s-3}} M^{\xi_A}) \\ \\ \widetilde{q}_i = F_i(A^{\frac{\eta_i}{3-2s}} M^{\eta_i}, A^{\frac{\xi_i}{2s-3}} M^{\xi_i}) \end{cases} \tag{60}$$

and if $s = 3/2$

$$\begin{cases} \widetilde{A} = -\ln M + F_A(A) G_A(MR) \\ \\ \widetilde{q}_i = F_i(A) G_i(MR) \end{cases} \tag{61}$$

where F_A, F_i, G_A and G_i are arbitrary functions and η_A, η_i, ξ_A and ξ_i integration constants.

These considerations show that the Noether Symmetries Approach can be applied to large classes of gravity theories. Up to now the Noether symmetries Approach has been worked out in the case of FRW - metric. In this paper, we have concentrated our attention to the development of the general formalism in the case of spherically symmetric spacetimes. Therefore the fact that, even in the case of a spherical symmetry, it is possible to achieve exact solutions seems to suggest that this technique can represent a paradigmatic approach to work out exact solutions in any theory of gravity. At this stage, the systematic search for exact solution is well beyond the aim of this paper. A more comprehensive analysis in this sense will be the argument of forthcoming studies. A final comment deserves the possible relevance of this approach for the above mentioned Birkhoff-Jensen theorem. The validity of such a theorem is crucial in every theory of gravity, due to the fact that it is directly related to the physical properties of self-gravitating systems (stability, stationarity, etc.). The results presented in this paper point out that it does not hold in general for the specific $f(R)$ theories considered. However, the above technique could be a good approach to select suitable classes of theories where such a theorem holds.

Appendix A. The $f(R)$ field equations in spherical symmetry

The field equations (Tab.1) in spherical symmetry, derived from the variational principle of the action (1), are

$$H_{00} = 2A^2B^2Mf + \{BMA'^2 - A[2BA'M' + M(2BA'' - A'B')]\}f_R +$$

$$+(-2A^2MB'R' + 4A^2BM'R' + 4A^2BMR'')f_{RR} +$$

$$+4A^2BMR'^2f_{RRR} = 0\,, \tag{A.1}$$

$$H_{rr} = 2A^2B^2M^2f + (BM^2A'^2 + AM^2A'B' + 2A^2MB'M' + 2A^2BM'^2 +$$

$$-2ABM^2A'' - 4A^2BMM'')f_R + (2ABM^2A'R' +$$

$$+4A^2BMM'R')f_{RR} = 0\,, \tag{A.2}$$

$$H_{\theta\theta} = 2AB^2Mf + (4AB^2 - BA'M' + AB'M' - 2ABM'')f_R +$$

$$+(2BMA'R' - 2AMB'R' + 2ABM'R' + 4ABMR'')f_{RR} +$$

$$+4ABMR'^2 f_{RRR} = 0\,, \tag{A.3}$$

$$H_{\varphi\varphi} = \sin^2\theta H_{\theta\theta} = 0\,. \tag{A.4}$$

The trace equation is

$$H = g^{\mu\nu}H_{\mu\nu} = 4AB^2Mf - 2AB^2MRf_R + 3(BMA'R' - AMB'R' +$$

$$+2ABM'R' + 2ABMR'')f_{RR} + 6ABMR'^2 f_{RRR} = 0 \tag{A.5}$$

Appendix B. The Noether vector

The system (44) comes out from the condition $L_{\mathbf{X}}\mathbf{L} = 0$ for the existence of the Noether symmetry. Considering the configuration space $\underline{q} = (A\,, M\,, R\,)$ and defining the Noether vector components $\underline{\alpha} = (\alpha_1\,, \alpha_2\,, \alpha_3)$, the system (44) assumes the explicit form

$$\xi\Big(\frac{\partial\alpha_2}{\partial A}f_R + M\frac{\partial\alpha_3}{\partial A}f_{RR}\Big) = 0 \tag{B.1}$$

$$\frac{A}{M}\Big[(2+MR)\alpha_3 f_{RR} - \frac{2\alpha_2}{M}f_R\Big]f_R +$$

$$+\xi\Big[\Big(\frac{\alpha_1}{M} + 2\frac{\partial\alpha_1}{\partial M} + \frac{2A}{M}\frac{\partial\alpha_2}{\partial M}\Big)f_R + A\Big(\frac{\alpha_3}{M} + 4\frac{\partial\alpha_3}{\partial M}\Big)f_{RR}\Big] = 0 \tag{B.2}$$

$$\xi\Big(M\frac{\partial\alpha_1}{\partial R} + 2A\frac{\partial\alpha_2}{\partial R}\Big)f_{RR} = 0 \tag{B.3}$$

$$\alpha_2(f - Rf_R)f_R - \xi\Big[\Big(\alpha_3 + M\frac{\partial\alpha_3}{\partial M} + 2A\frac{\partial\alpha_3}{\partial A}\Big)f_{RR} +$$

$$+\Big(\frac{\partial\alpha_2}{\partial M} + \frac{\partial\alpha_1}{\partial A} + \frac{A}{M}\frac{\partial\alpha_2}{\partial A}\Big)f_R\Big\} = 0 \tag{B.4}$$

$$[M(2+MR)\alpha_3 f_{RR} - 2\alpha_2 f_R]f_{RR} + \xi\Big[f_R\frac{\partial\alpha_2}{\partial R} +$$

$$+\Big(2\alpha_2 + M\frac{\partial\alpha_1}{\partial A} + 2A\frac{\partial\alpha_2}{\partial A} + M\frac{\partial\alpha_3}{\partial R}\Big)\alpha_3 f_{RR} + Mf_{RRR}\Big\} = 0 \tag{B.5}$$

$$2A[(2+MR)\alpha_3 f_{RR} - (f - Rf_R)\alpha_2]f_{RR} + \xi\Big[\Big(\frac{\partial\alpha_1}{\partial R} + \frac{A}{M}\frac{\partial\alpha_2}{\partial R}\Big)f_R +$$

$$+\Big(2\alpha_1 + 2A\frac{\partial\alpha_3}{\partial R} + M\frac{\partial\alpha_1}{\partial M} + 2A\frac{\partial\alpha_2}{\partial M}\Big)f_{RR} + 2A\alpha_3 f_{RRR}\Big] = 0 \qquad \text{(B.6)}$$

with the condition $\xi = (2+MR)f_R - Mf \neq 0$, otherwise the Hessian of Lagrangian **L** (18) is vanishing.

References

[1] Starobinsky A.A. 1980, Phys. Lett. B, 91, 99

[2] Kerner R. 1982, Gen. Rel. Grav., 14, 453

[3] Capozziello S. 2002, Int. J. Mod. Phys. D, 11, 483.
Capozziello S., Carloni S., Troisi A. 2003, Rec. Res. Dev. in Astron. and Astroph., 1, 1, (arXiv : astro - ph/0303041).
Odintsov S.D., Nojiri S. 2003, Phys. Lett. B, 576, 5
Capozziello S., Cardone V.F., Carloni S., Troisi A. 2003, Int. J. Mod. Phys. D, 12, 1969.
Carroll S.M., Duvvuri V., Trodden M., Turner M. 2004, Phys. Rev. D, 70, 043528.
Allemandi G., Borowiec A., Francaviglia M. 2004, Phys. Rev. D, 70, 103503.
Nojiri S. and Odintsov S.D. 2004, Gen. Rel. Grav. 36, 1765.
Cognola G., Elizalde E., Nojiri S., S.D. Odintsov, Zerbini S. 2005, JCAP, 010.

[4] Capozziello S., Cardone V.F., Carloni S., Troisi A. 2004, Phys. Lett. A, 326, 292.
Capozziello S., Cardone V.F., Troisi A., 2007 *Mon. Not. R. Ast. Soc.* **375**, 1423.

[5] Capozziello S., Cardone V.F., Troisi A., 2006, JCAP **08**, 001.

[6] Navarro I., Van Acoleyen I., arXiv:astro-ph/0611127.

[7] Bean R., *et al.*, arXiv:astro-ph/0611321.

[8] Li B., Barrow J., arXiv:astro-ph/0701111.

[9] Capozziello S., Cardone V.F., Troisi A. 2005, Phys. Rev. D, 71, 043503

[10] Capozziello S., Nojiri S., Odintsov S.D., Troisi A., 2006, Phys. Lett. B 639, 135.

[11] Amendola L., Polarski D., Tsujikawa S., 2006, arXiv:astro-ph/0603703,
Amendola L., Gannouji R., Polarski D., Tsujikawa S., 2006, arXiv:astro-ph/0612180.

[12] Capozziello S., Troisi A. 2005, Phys. Rev. D, 72, 044022

[13] Allemandi G., Francaviglia M., Ruggiero M.L., Tartaglia A., 2005, Gen. Rel. Grav. 37, 1891;
Ruggiero M.L., Iorio L., 2007, JCAP 0701, 010.

[14] Faulkner T., Tegmark M., Bunn E.F., Mao Y., arXiv:astro-ph/0612569 .

[15] Olmo G., 2005, Phys.Rev.D 72, 083505.

[16] Erickcek A.L., Smith T. L., Kamionkowski M., 2006, Phys.Rev. D74, 121501.

[17] Faraoni V., Nadeau S., 2007, Phys. Rev. D 75, 023501.

[18] Stelle K. 1978, Gen. Rel. Grav., 9, 353.

[19] Multamäki T., Vilja I. *Phys. Rev.* **D 74**, 064022 (2006).

[20] Multamäki T., Vilja I. (2006) arXiv:astro-ph/0612775 (2006).

[21] de Ritis R., Marmo G., Platania G., Rubano C., Scudellaro P., Stornaiolo C., 1990, *Phys. Rev.* **42D**, 1091.

[22] Capozziello S. and de Ritis R., 1993, *Phys. Lett.* **A 177**, 1.
Capozziello S., de Ritis R., Rubano C., Scudellaro P., 1996, La Riv. del Nuovo Cimento, **19**, 4, 1.

[23] Hawking S.W. and Ellis G.F.R., 1973 *The large scale structure of space-yime*, Cambridge Univ. Press, Cambridge, UK.

[24] Clifton T., PhD - thesis, arXiv:gr-qc/0610071.

[25] Neville D., 1980, Phys. Rev. **D 21**, 10.
[26] Ramaswamy S. and Yasskin P.B. 1979, Phys. Rev. **D 19**, 2264.
[27] D. E. Barraco and V. H. Hamity, Phys. Rev., 1998, **D 57**, 954;
D. E. Barraco D.E. and Hamity V.H., 2000, Phys. Rev. **D 62**, 044027.
[28] Clifton T. and Barrow J.D., 2005, Phys. Rev. D **72**, 103005,
Clifton T. and Barrow J.D., 2006, *Class. Quantum Grav.* **23**,2951;
Clifton T. and Barrow J.D., 2006, *Class. Quantum Grav.* **23**, L1.
[29] Capozziello S. and de Ritis R., 1994, *Class. Quantum Grav.* **11**, 107.
[30] Sanyal A.K., Rubano C., Piedipalumbo E., 2003, *Gen. Relativ. Grav.* **35**, 1617.
Kamilya S., Modak B., and Biswas S., 2004, *Gen. Relativ. Grav.* **36**, 661.
[31] Capozziello S. and Lambiase G., 2000, *Gen. Relativ. Grav.* **32**, 295.
[32] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer–Verlag, Berlin (1978).
[33] G. Marmo, E.J. Saletan, A. Simoni and B. Vitale, *Dynamical Systems. A Differential Geometric Approach to Symmetry and Reduction*, Wiley, New York (1985).
[34] Morandi G., Ferrario C., Lo Vecchio G., Marmo G., Rubano C., 1990, Phys. Rep. 188, 149